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Discrete actions on nilpotent Lie groups and negatively curved spaces

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Abstract

The aim of this paper is to study dynamics of a discrete isometry group action in a pinched Hadamard manifold nearby its parabolic fixed points. Due to Margulis Lemma, such an action on corresponding horospheres is virtually nilpotent, so we solve the problem by establishing a structural theorem for discrete groups acting on connected nilpotent Lie groups. As applications, we show that parabolic fixed points of a discrete isometry group cannot be conical limit points, that the fundamental groups of geometrically finite orbifolds with pinched negative sectional curvature are finitely presented, and the orbifolds themselves are topologically finite.

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1. Introduction

A Hadamard manifold is a complete, simply connected Riemannian manifold of nonpositive curvature. By a pinched Hadamard manifold, we shall mean a Hadamard manifold whose all sectional curvatures K lie between two negative constants, $-b^2 \leqslant K \leqslant -a^2$, where $0 < a < \infty$ and $0 < b < \infty$.

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The main aim of this paper is to describe a discrete group action on pinched Hadamard manifolds, in particular on symmetric spaces of rank 1 with negative curvature, nearby its parabolic fixed points. All those spaces are foliated by horospheres centered at a given point $p \in X(\infty)$ at infinity (i.e., by level surfaces of a Busemann function, cf. [12,19]). The most important case (for understanding of general situation) is that of a discrete (parabolic) group $\Gamma \subset \operatorname{Isom} X$ that fixes the point p and preserves setwise each of those horospheres. In this case, by applying the Margulis Lemma [12,33], it follows that a discrete parabolic group Γ is virtually nilpotent. Furthermore, at least in symmetric spaces of rank 1 with negative curvature (that is hyperbolic spaces—either real, complex, quaternionic or octonionic ones), all those horospheres can be identified with a connected simply connected Lie group $\mathcal{N}$ and our discrete group Γ isometrically acts on $\mathcal{N}$ as a subgroup $\Gamma \subset \mathcal{N} \rtimes C$ where C is a compact group of automorphisms of $\mathcal{N}$. In the case of real hyperbolic spaces (of constant negative curvature), horospheres are flat, and discrete Euclidean isometry group actions are described by the Bieberbach theorem, see [42]. However in the other symmetric spaces of rank 1 this is no longer true. In these spaces horospheres may be represented as non-Abelian nilpotent Lie groups with a left invariant metric, and therefore have curvatures of both signs [41]. Here we come to the main result of this paper, the following structural theorem:

Theorem 4.1. *Let $\mathcal{N}$ be a connected, simply connected nilpotent Lie group, C be a compact group of automorphisms of $\mathcal{N}$, and $\Gamma \subset \mathcal{N} \rtimes C$ be a discrete subgroup. Then there exist a connected Lie subgroup $\mathcal{N}_\Gamma$ of $\mathcal{N}$ and a finite index subgroup Γ^* of Γ with the following properties*:

(1) *There exists $b \in \mathcal{N}$ such that $b\Gamma b^{-1}$ preserves $\mathcal{N}_\Gamma$;*
(2) *$\mathcal{N}_\Gamma / b\Gamma b^{-1}$ is compact;*
(3) *$b\Gamma^* b^{-1}$ acts on $\mathcal{N}_\Gamma$ by left translations, and this action is free.*

Here the compactness condition on the group C of automorphisms of $\mathcal{N}$ is essential. The situation when the group C may be noncompact is completely different. For instance, Margulis [34,35] constructed discrete subgroups of $R^3 \rtimes \mathrm{SO}(2,1)$ which are nonabelian free groups, whereas in the compact case any discrete subgroup of $\mathcal{N} \rtimes C$ must be virtually nilpotent (see Remark 4.2), which resembles Gromov's almost flat manifolds [17,23]. On the other hand, when the group C is compact, there exists a left invariant metric on $\mathcal{N}$ such that $\mathcal{N} \rtimes C$ acts on $\mathcal{N}$ as a group of isometries. So any discrete subgroup of $\mathcal{N} \rtimes C$ can be viewed as a discrete isometry group of $\mathcal{N}$ with respect to some left invariant metric. We remark that Theorem 4.1 advances a result by Louis Auslander [2] who proved the claims (1) and (2) of Theorem 4.1 only for a finite index subgroup of a given discrete group Γ.

A motivation for our study comes from an attempt to understand parabolic (the so-called "thin") ends of negatively curved manifolds, as well as the geometry and topology of geometrically finite pinched negatively curved manifolds, see [3,4,7,10,14,15]. The concept of geometrical finiteness first arose in the context of (real) hyperbolic 3-manifolds. Its original definition (due to Ahlfors [1]) came from an assumption that such a geometrically finite real hyperbolic manifold M may be decomposed into a cell by cutting along a finite number of its totally geodesic hypersurfaces. Since that time, other definitions of geometrical finiteness have been given by Marden [32], Beardon and Maskit [13], and Thurston [39], and the notion has become central to the study of real hyperbolic manifolds. Though other pinched Hadamard manifolds may not have totally geodesic hypersurfaces, the other definitions of geometrical finiteness work in the case of variable negative curvature as well, see [5,6,10,15]. Our previous paper [10] represents a successful step towards studying geometrical finiteness in variable curvature in the case of complex hyperbolic manifolds, on the base of a structural theorem for discrete isometric actions on the Heisenberg groups, a predecessor of Theorem 4.1. However the proof of the structural theorem in the case of Heisenberg groups extends the (Euclidean) argument by Wolf [42] and specifically uses the Heisenberg geometry, so it does not work in general nilpotent groups. Here our proof of Theorem 4.1 uses different algebraic ideas.

As the first applications of our structural Theorem 4.1, we basically present two results (another applications will appear elsewhere, see [8]). The first one (Proposition 5.3) answers a question on dynamics of a discrete isometry group action nearby its limit points, which was left open for variable negative curvature spaces. Namely, it distinguishes two types of limit points of a discrete group $G \subset \operatorname{Isom} X$ acting on a pinched Hadamard manifold X with N-property (i.e., whose horospheres are connected nilpotent Lie groups with compact automorphism groups), in particular on symmetric spaces of rank 1. Namely it shows that parabolic fixed points of such a discrete group G cannot be its conical limit points, i.e., such points $z \in X(\infty)$ that for some (and hence every) geodesic ray ℓ in X ending at z, there is a compact set $K \subset X$ such that the subset $\{g \in G\colon\ g(\ell) \cap K \neq \emptyset\}$ is infinite. Such a dichotomy has been recently proved only in the case of real hyperbolic spaces (of constant curvature) by Susskind and Swarup [38] and independently, from a dynamical point of view, by Starkov [37].

The second our application (Theorem 6.5 and Corollary 6.6) of the structural Theorem 4.1 answers another open question (formulated as a conjecture in [15, p. 230]). Namely, it shows that discrete parabolic groups Γ isometrically acting on a connected Lie groups $\mathcal{N}$ with a compact automorphism group, as well as geometrically finite discrete groups $G \subset \operatorname{Isom} X$ acting on the corresponding pinched Hadamard manifold with N-property (including symmetric spaces of rank 1) are finitely presented, and the corresponding quotient orbifolds are topologically finite. Previously, it was known for constant negative curvature. For pinched Hadamard manifolds with various negative curvature, Bowditch [14,15] proved that such groups are finitely generated. The answer in the case of Heisenberg groups and complex hyperbolic manifolds has been earlier given by the authors in [10,11].

2. Horospherical coordinates on pinched Hadamard manifolds

Let X be a pinched Hadamard manifold with a distance function d and sectional curvature K, $-b^2 \leqslant K \leqslant -a^2$. Its infinity $X(\infty) = \partial X$ is homeomorphic to the sphere S^{n-1} where $n = \dim X$, and the isometry group $\operatorname{Isom} X$ acts at infinity as a convergence group [21]. So one can classify isometries $g \in \operatorname{Isom} X$ in the following three types. If g fixes a point in X, it is called *elliptic*. If g has exactly one fixed point, and it lies in ∂X, g is called *parabolic*. If g has exactly two fixed points, and they lie in ∂X, g is called *loxodromic*. These three types exhaust all the possibilities.

A subgroup $G \subset \operatorname{Isom} X$ is called *discrete* if it is a discrete subset of $\operatorname{Isom} X$. The *limit set* $\Lambda(G) \subset \partial X$ of a discrete group G is the set of accumulation points of (any) orbit $G(y),\ y \in X$. The complement of $\Lambda(G)$ in ∂X is called the *discontinuity set* $\Omega(G)$. A discrete group G is called *elementary* if its limit set $\Lambda(G)$ consists of at most two points. An infinite discrete group G is called *parabolic* if it has exactly one fixed point $\{p\} = \operatorname{fix}(G)$; then $\Lambda(G) = \{p\} = \operatorname{fix}(G)$, and G consists of either parabolic or elliptic elements because a discrete group of isometries of a negatively curved space has no loxodromic and parabolic elements with a common fixed point, see [19,21].

In order to study discrete parabolic subgroups in $\operatorname{Isom} X$, it is convenient to view X from a fixed point $q_\infty \in \partial X$. Then subspaces issuing from this point q_∞ can be used to define "horospherical coordinates" on X and its boundary $\partial X = X(\infty)$ at infinity. We obtain the "upper half-space model" for X as follows, similarly to the Poincaré and Goldman–Parker [22] half-space models for real and complex hyperbolic spaces, correspondingly.

Fixing a point $q \in \partial X$ at infinity of X, one can define two foliations of X. The first one consists of geodesics ending at q, $\ell : (-\infty, +\infty) \longrightarrow X$ where $\ell(t)$ converges to $q \in \partial X$ as t goes to ∞. The second foliation consists of horospheres centered at q. To define horospheres, we can use the Busemann function $F = \lim F_t$, where $F_t(x) = d(x, \ell(t)) - t$ with $\ell(t_0) = x \in X$, see [12,19]. Then the horospheres centered at q are the level surfaces of the Busemann function F; they bound horoballs $F^{-1}(t_0, +\infty)$ centered at q. Due to [25], the Busemann functions (and therefore horospheres) are at least C^2-smooth. Hence the notions of distance and geodesic curves are defined with respect to the induced metric which decreases exponentially as x approaches $q \in \partial X$, i.e., $t \to +\infty$, see [25] for estimates which depend on the pinching constants $a, b > 0$. As level surfaces of a Busemann function, horospheres are closed and therefore complete, and one always have minimal geodesics joining two points.

As it was shown in [25], the geometry of horospheres in X may be closely compared with that in the spaces of constant negatives curvature $-a^2$ and $-b^2$, respectively. In particular, for two asymptotic geodesic rays ℓ and ℓ' approaching $q \in \partial X$ from two points x and x' on the same horosphere, with a horospherical distance R_0 between them, we have:

$$\left(\frac{2}{b} \operatorname{arcsinh} \frac{2}{b} R_0\right) \mathrm{e}^{-bt} \leqslant d\big(\ell(t), \ell'(t)\big) \leqslant R_0 \mathrm{e}^{-at}. \tag{2.1}$$

Even for geodesic triangles with vertices $x, y \in X$ and with one vertex q at infinity, where the lengths of the infinite sides are not defined, one can however measure their difference. Namely, the (oriented) difference $h = F(x) - F(y)$ is independent of the choice of a Busemann function F at q and, for the corresponding (unique) triangles and the differences h_a and h_b in the spaces of constant curvatures $-a^2$ and $-b^2$, the following comparison holds [25, Proposition 4.4]:

$$h_a \leqslant h \leqslant h_b. \tag{2.2}$$

From now on, we should assume an additional condition (N) on our pinched Hadamard manifolds X:

(N) The parabolic subgroup of $\operatorname{Isom} X$ that fixes a point $q \in \partial X$ at infinity acts transitively on each horosphere $X_t \subset X$ centered at q, which can be identified with a connected, simply connected nilpotent Lie group $\mathcal{N}$ with a compact group C of automorphisms.

Obviously, symmetric spaces of rank 1 with negative curvature are in our class of pinched Hadamard manifolds with N-property [26]. As we observe in Remark 4.2, in this case each horosphere $X_t \subset X$, $X_t \cong \mathcal{N}$, has a left invariant metric preserved by the action of the parabolic group fixing q and isomorphic to $\mathcal{N} \rtimes C$. With respect to this metric, those various horospheres isometrically correspond via the geodesic perspective from q at infinity, $X_t \to X_s$. Furthermore, taking the composition $\exp \circ F : X \to (0, \infty)$ of the exponent and a Busemann function F at q, we may identify

$$\overline{X} \backslash \{q\} = X \cup \partial X \backslash \{q\} \to \mathcal{N} \times [0, \infty), \tag{2.3}$$

and call this identification the "*upper half-space model*" for our pinched Hadamard manifold X that satisfies the condition (N). The identification (2.3) and the standard coordinates on the nilpotent (Carnot) group $\mathcal{N}$ give us *horospherical coordinates* on X.

In particular, for rank one symmetric spaces X, that is the hyperbolic spaces $\mathbb{H}^n_{\mathbb{F}}$ associated with the real, complex, quaternionic or Cayley numbers $\mathbb{F}$, the sphere at infinity ∂X can be identified with one point compactification of the nilpotent group $\mathcal{N}$ in the Iwasawa decomposition of $\operatorname{Isom} X = \mathcal{KAN}$. In its turn, the nilpotent group $\mathcal{N}$ can be identified with the product $\mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}$ equipped with the operations:

$$(\xi, v)\cdot(\xi', v') = \bigl(\xi + \xi', v + v' + 2\operatorname{Im}\langle\xi, \xi'\rangle\bigr) \quad \text{and} \quad (\xi, v)^{-1} = (-\xi, -v), \tag{2.4}$$

where $\langle , \rangle$ is the standard Hermitian product in $\mathbb{F}^{n-1}$, $\langle z, w\rangle = \sum z_i \overline{w_i}$. The group $\mathcal{N}$ is a 2-step nilpotent Carnot group with center $\{0\} \times \operatorname{Im}\mathbb{F} \subset \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}$, and acts on itself by the left translations $T_h(g) = h \cdot g,\ h, g \in \mathcal{N}$.

In horospherical coordinates of a given rank one symmetric spaces $X = \mathbb{H}^n_{\mathbb{F}}$, we identify $\overline{X}\backslash\{\infty\}$ with the product $\mathcal{N} \times [0, \infty)$, that is with $\mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \times [0, \infty)$. So in these coordinates, the left action (2.4) of the Carnot group $\mathcal{N}$ on itself extends to an isometric action (Carnot translation) on the $\mathbb{F}$-hyperbolic space in the following form:

$$T_{(\xi_0, v_0)} : (\xi, v, u) \mapsto \bigl(\xi_0 + \xi\ , v_0 + v + 2\operatorname{Im}\langle\xi_0, \xi\rangle, u\bigr), \tag{2.5}$$

where $(\xi, v, u) \in \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \times [0, \infty)$.

There are a natural norm and its induced distance on the Carnot group $\mathcal{N} = \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F}$, which are known in the case of the Heisenberg group (when $\mathbb{F} = \mathbb{C}$) as the Cygan's norm and distance, see [18,22] and, in gauge terms, [29,30] and Remark 6.4. Namely, it assigns to $(\xi, v) \in \mathcal{N}$ the following non-negative real number:

$$\bigl|(\xi, v)\bigr|_c = \bigl(|\xi|^4 + |v|^2\bigr)^{1/4} = \bigl|\bigl(|\xi|^2 - v\bigr)\bigr|^{1/2}, \tag{2.6}$$

where $|.|$ is the norm in $\mathbb{F}$.

To check that (2.6) really defines a norm, it is essential to verify its triangle inequality:

$$\bigl|(\xi', v')^{-1} \cdot (\xi, v)\bigr|_c \leqslant \bigl|(\xi', v')\bigr|_c + \bigl|(\xi, v)\bigr|_c, \tag{2.7}$$

which can be done directly.

In particular, for quaternionic case when a quaternion $z = (a + \mathrm{i}b + \mathrm{j}c + \mathrm{k}d) \in H$ has its imaginary part $\operatorname{Im} z$ equal to $\mathrm{i}b + \mathrm{j}c + \mathrm{k}d$, $a, b, c, d \in \mathbb{R}$, one has such a triangle inequality, [28]:

$$\begin{aligned}
\bigl|(\xi', v')^{-1} \cdot (\xi, v)\bigr|_c^2 &= \bigl|(-\xi', -v') \cdot (\xi, v)\bigr|_c^2 \\
&= \bigl|\bigl(\xi - \xi', v - v' + 2\operatorname{Im}\langle -\xi, \xi\rangle\bigr)\bigr|_c^2 \\
&= \bigl||\xi - \xi'|^2 + v - v' - 2\operatorname{Im}\langle\xi', \xi\rangle\bigr| \\
&= \bigl|\langle\xi - \xi', \xi - \xi'\rangle + v - v' - \xi'\overline{\xi} + \xi\overline{\xi'}\bigr| \\
&= \bigl||\xi|^2 + |\xi'|^2 - \xi\overline{\xi'} - \xi'\overline{\xi} + v - v' - \xi'\overline{\xi} + \xi\overline{\xi'}\bigr| \\
&= \bigl||\xi|^2 + v\bigr| + \bigl||\xi'|^2 - v'\bigr| + 2|\xi'\overline{\xi}| \\
&\leqslant \bigl|(\xi, v)\bigr|^2 + \bigl|(\xi', v')\bigr|^2 + 2\bigl|(\xi', v')\bigr| \cdot \bigl|(\xi, v)\bigr| \\
&= \bigl(\bigl|(\xi, v)\bigr| + \bigl|(\xi', v')\bigr|\bigr)^2,
\end{aligned} \tag{2.8}$$

where (ξ', v') and (ξ, v) are arbitrary elements of the quaternionic Carnot group $\mathcal{N} = H^{n-1} \times \operatorname{Im} H$.

Using horospherical coordinates on $\overline{X}\backslash\{\infty\} = \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \times [0, \infty)$, we can extend the norm (2.6) to a norm on the $\mathbb{F}$-hyperbolic space X:

$$\left|(\xi, v, u)\right|_c = \left|\left(|\xi|^2 + u - v\right)\right|^{1/2}. \tag{2.9}$$

This norm then gives rise to a distance on the rank one symmetric space $X = \mathbb{H}^n_{\mathbb{F}}$ (on its upper half-space model $\mathbb{H}^n_{\mathbb{F}} = \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \times (0, \infty)$), which we still call the Cygan distance:

$$\rho_c : \left(\mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \times (0, \infty)\right) \times \left(\mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \times (0, \infty)\right) \to \mathbb{R},$$

$$\begin{aligned}\rho_c\big((\xi, v, u), (\xi', v', u')\big) &= \left|\left(T^{-1}_{(\xi', v')}(\xi, v), |u - u'|\right)\right|_c \\ &= \left||\xi - \xi'|^2 + |u - u'| - \left(v - v' + 2\operatorname{Im}\langle \xi, \xi'\rangle\right)\right|^{1/2}.\end{aligned} \tag{2.10}$$

In fact, it follows directly from the definition that Carnot translations and rotations are isometries of $\mathbb{H}^n_{\mathbb{F}} = \mathbb{F}^{n-1} \times \operatorname{Im}\mathbb{F} \times (0, \infty)$ with respect to the Cygan distance ρ_c. Moreover, the restrictions of this distance to different horospheres centered at ∞ are the same, so Cygan distance plays the same role as Euclidean distance does on the upper half-space model for the real hyperbolic space $\mathbb{H}^n$.

Also we note that the relevant geometry on each horosphere $X_t \cong \mathcal{N}$ is the Carnot–Carathéodory geometry (cf. Gromov [24,36]) induced by the negatively curved metric of X. The geodesic perspective from q_∞ defines conformal maps between horospheres X_t and X_s which extends to conformal maps between the one-point compactifications $X_t \cup \infty$ homeomorphic to spheres S^{n-1}. In the limit, the induced metrics on horospheres fail to converge but the Carnot–Carathéodory structure remains fixed. In this way, the negatively curved geometry on X induces the Carnot–Carathéodory geometry on the sphere at infinity $\partial X \approx S^{n-1}$, naturally identified with the one-point compactification of the nilpotent (Carnot) group $\mathcal{N}$ (for symmetric spaces of rank 1, see Pansu [36]).

3. Group cohomology and group extensions

In this section, we review some facts about group cohomology and group extensions. The main references are [16] and [31].

An extension of a group G by a group N is a short exact sequence of groups

$$1 \longrightarrow N \xrightarrow{\ i\ } E \xrightarrow{\ \pi\ } G \longrightarrow 1. \tag{$*$}$$

We say that this exact sequence $(*)$ splits if there is a homomorphism $s : G \to E$ such that $\pi s = id_G$. Another extension $1 \to N \to E' \to G \to 1$ of the group G by N is said to be equivalent to the first one (*) if there is an isomorphism $E \to E'$ that makes the diagram

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & N \arrow[r] & E' \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

commute.

Note that an extension $1 \to N \to E \to G \to 1$ gives rise to a commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\Psi"] & 1 \\
1 \arrow[r] & \mathrm{Inn}(N) \arrow[r] & \mathrm{Aut}(N) \arrow[r] & \mathrm{Out}(N) \arrow[r] & 1
\end{tikzcd}$$

where Ψ is induced by the conjugation of E on N. Now we fix a homomorphism $\Psi : G \to \mathrm{Out}(N)$ and let $\varepsilon(G, N, \Psi)$ be the set of equivalence classes of extensions giving rise to Ψ. Observe that there is a semi-direct product $N \rtimes G$ in $\varepsilon(G, N, \Psi)$ if and only if Ψ lifts to a homomorphism $G \to \mathrm{Aut}(N)$. We will need the following fact, see [20]:

Theorem 3.1. *Either $\varepsilon(G, N, \Psi) = \emptyset$ or there is a bijection between $\varepsilon(G, N, \Psi)$ and $H^2(G, C)$, where C is the center of N.*

As a corollary of this theorem, we obtain the following result:

Proposition 3.2. *Let $\mathcal{N}$ be a connected, simply connected nilpotent Lie group, and Φ a finite group. Then any extension of Φ by $\mathcal{N}$ splits.*

Proof. We induct on the nilpotency $\mathrm{nil}(\mathcal{N})$ of the group $\mathcal{N}$. When $\mathrm{nil}(\mathcal{N}) = 1$, our claim holds because $\mathcal{N}$ is an Euclidean space and the extensions of Φ by $\mathcal{N}$ are classified by $H^2(\Phi, \mathcal{N})$, which is trivial.

Suppose the theorem assertion is true for all groups $\mathcal{N}$ with $\mathrm{nil}(\mathcal{N}) < k + 1$ and let us take a group $\mathcal{N}$ with $\mathrm{nil}(\mathcal{N}) = k + 1$. Any extension $1 \to \mathcal{N} \to E \to \Phi \to 1$ induces a commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{N} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \Phi \arrow[r] \arrow[d, "\Psi"] & 1 \\
1 \arrow[r] & \mathrm{Inn}(\mathcal{N}) \arrow[r] & \mathrm{Aut}(\mathcal{N}) \arrow[r, "\pi'"] & \mathrm{Out}(\mathcal{N}) \arrow[r] & 1.
\end{tikzcd}$$

This diagram gives rise to an exact sequence

$$1 \longrightarrow \mathrm{Inn}(\mathcal{N}) \longrightarrow (\pi')^{-1}(\Psi(\Phi)) \xrightarrow{\pi'} \Psi(\Phi) \longrightarrow 1.$$

Since $\mathrm{Inn}(\mathcal{N}) \cong \mathcal{N}/C(\mathcal{N})$ is a connected, simply connected nilpotent Lie group with nilpotency k, the above exact sequence splits according to the induction hypothesis. So there exists a homomorphism $s : \Psi(\Phi) \to (\pi')^{-1}(\Psi(\Phi))$ such that $\pi' \circ s = id_{\Psi(\Phi)}$. Thus $\pi' \circ s \circ \Psi = \Psi$, and $s \circ \Psi$ is a lifting of Ψ. This implies that $\varepsilon(\Phi, \mathcal{N}, \Psi)$ contains a semi-direct product. Due to Theorem 3.1, $\varepsilon(\Phi, \mathcal{N}, \Psi)$ is in one-to-one correspondence with $H^2(\Phi, C(\mathcal{N}))$. On the other hand, $H^2(\Phi, C(\mathcal{N})) = 0$ because Φ is finite and $C(\mathcal{N})$ is a Euclidean space. It follows that $\varepsilon(\Phi, \mathcal{N}, \Psi)$ contains the only one element which is a semi-direct product. Since the exact sequence $1 \to \mathcal{N} \to E \to \Phi \to 1$ does represent an element in $\varepsilon(\Phi, \mathcal{N}, \Psi)$, it is a semi-direct product. Therefore any extension of Φ by $\mathcal{N}$ splits. □

Next, we introduce the first cohomology with nonabelian coefficients. Let G and N be two groups, and let $\phi : G \to \mathrm{Aut}(N)$ be a homomorphism. A map $f : G \to N$ is called a 1-cocycle (with respect to ϕ) if

$$f(xy) = \phi(x)\big(f(y)\big) \cdot f(x)$$

for all $x, y \in G$. Two 1-cocycles f_0 and f_1 are said to be equivalent if there is an element $b \in N$ such that

$$f_1(x) = \phi(x)\bigl(b^{-1}\bigr) \cdot f_0(x) \cdot b$$

for all $x \in G$. So we have the first cohomology $H^1(G, N)$ as the set of equivalence classes of 1-cocycles of G into N.

Due to a Lee and Raymond's result [31], for any finite group Φ and a connected, simply connected nilpotent Lie group $\mathcal{N}$, the first cohomology vanishes $H^1(\Phi, \mathcal{N}) = 0$. On the other hand, since the trivial homomorphism from Φ to $\mathcal{N}$ is a 1-cocycle, for any 1-cocycle $f : \Phi \to \mathcal{N}$ (with respect to a homomorphism $\phi : \Phi \to \mathrm{Aut}(\mathcal{N})$) there is an element $b \in \mathcal{N}$ such that

$$f(x) = \phi(x)\bigl(b^{-1}\bigr) \cdot b \quad \text{for all } x \in \Phi.$$

Finally, we formulate a result [40] about automorphisms θ of a nilpotent Lie group $\mathcal{N}$. Each such an automorphism θ induces an automorphism $\tilde{\theta}$ of the Lie algebra $\mathfrak{n}$ of the group $\mathcal{N}$. An automorphism θ is called semi-simple if the induced automorphism $\tilde{\theta}$ is semi-simple, that is, for each subspace L of the Lie algebra $\mathfrak{n}$ invariant under $\tilde{\theta}$ there is a complementary invariant subspace. In particular, an automorphism θ of a nilpotent Lie group $\mathcal{N}$ is semi-simple if it belongs to a compact group of automorphisms of $\mathcal{N}$.

Due to Lemma 5.1 in [40], we have:

Lemma 3.3. *Let θ be a semi-simple automorphism of a connected, simply connected nilpotent Lie group $\mathcal{N}$. For elements $v, w \in \mathcal{N}$, if both v and $\theta(w)vw^{-1}$ are fixed by θ, then $\theta(w) = w$.*

4. Discrete isometry actions on nilpotent Lie groups

Here we shall prove the following structural theorem describing discrete isometry group actions on nilpotent Lie groups:

Theorem 4.1. *Let $\mathcal{N}$ be a connected, simply connected nilpotent Lie group, C be a compact group of automorphisms of $\mathcal{N}$, and $\Gamma \subset \mathcal{N} \rtimes C$ be a discrete subgroup. Then there exist a connected Lie subgroup $\mathcal{N}_\Gamma$ of $\mathcal{N}$ and a finite index subgroup Γ^* of Γ with the following properties*:

(1) *There exists $b \in \mathcal{N}$ such that $b\Gamma b^{-1}$ preserves $\mathcal{N}_\Gamma$*;
(2) *$\mathcal{N}_\Gamma / b\Gamma b^{-1}$ is compact*;
(3) *$b\Gamma^* b^{-1}$ acts on $\mathcal{N}_\Gamma$ by left translations, and this action is free.*

Remark 4.2. It follows from the claim (3) in Theorem 4.1 that Γ has a finite index subgroup Γ^* which is isomorphic to a discrete subgroup of the nilpotent Lie group $\mathcal{N}_\Gamma \subset \mathcal{N}$. Also, due the above observations on semi-simple automorphisms of $\mathcal{N}$ from a compact group C, there exists a left invariant metric on the group $\mathcal{N}$ such that $\mathcal{N} \rtimes C$ acts on $\mathcal{N}$ as a group of isometries. So any discrete subgroup of $\mathcal{N} \rtimes C$ can be viewed as a discrete isometry group of $\mathcal{N}$ with respect to some left invariant metric.

Proof of Theorem 4.1. We start with the first assertion of Theorem 4.1. Let $p : \Gamma \to C$ be the composition of the inclusion $\Gamma \subset \mathcal{N} \rtimes C$ and the projection $\mathcal{N} \rtimes C \to C$.

Lemma 4.3. *By replacing $\mathcal{N}$ with a connected Lie subgroup, we can assume that $p(\Gamma)$ is finite.*

Proof. Let G be the identity component of $\overline{\Gamma\mathcal{N}}$ and let $\Gamma_1 = G \cap \Gamma$. Due to compactness of C, the identity component G has a finite index in $\overline{\Gamma\mathcal{N}}$, hence Γ_1 has a finite index in Γ. Let W be the analytic subgroup of $\mathcal{N}$ left pointwise fixed by $p(\Gamma_1)$. Due to Auslander's [2] proof, there exists an element $b_0 \in \mathcal{N}$ such that for all elements $\gamma = (w, c) \in b_0\Gamma_1 b_0^{-1} \subset \mathcal{N} \rtimes C$, their parts w lie in W. Now we can consider the conjugation $b_0\Gamma b_0^{-1}$ instead of the group Γ. To avoid too much notation, we still denote $b_0\Gamma b_0^{-1}$ and $b_0\Gamma_1 b_0^{-1}$ by Γ and Γ_1, respectively.

Since Γ_1 has finite index in Γ, we may assume that Γ_1 is a normal subgroup in Γ. Now we shall show that for any element $\gamma = (a, A)$ in Γ, the automorphism A preserves W, $A(W) = W$, and the element a lies in W.

Let $\gamma_0 = (a_0, A_0)$ be an arbitrary element in Γ_1. Since Γ_1 is normal in Γ, the conjugation

$$\gamma\gamma_0\gamma^{-1} = \bigl(a \cdot A(a_0) \cdot AA_0A^{-1}\bigl(a^{-1}\bigr), AA_0A^{-1}\bigr)$$

belongs to Γ_1. Therefore the element $a \cdot A(a_0) \cdot AA_0A^{-1}(a^{-1})$ lies in W, and $AA_0A^{-1}(w) = w$ for any $w \in W$. So $A_0(A^{-1}(w)) = A^{-1}(w)$ for all $A_0 \in p(\Gamma_1)$, which implies that $A^{-1}(w) \in W$ and $A(W) = W$. Since $a \cdot A(a_0) \cdot AA_0A^{-1}(a^{-1})$ lies in W, we have that $AA_0A^{-1}(a) \cdot A(a_0^{-1}) \cdot a^{-1}$ belongs to W. As both elements $A(a_0^{-1}) \in W$ and $AA_0A^{-1}(a) \cdot A(a_0^{-1}) \cdot a^{-1}$ are fixed by the automorphism AA_0A^{-1}, Lemma 3.3 implies that $AA_0A^{-1}(a) = a$ for all $A_0 \in p(\Gamma_1)$. It follows from the definition of W that $a \in W$.

Now the group Γ preserves the subgroup $W \subset \mathcal{N}$, and we can replace $\mathcal{N}$ with W. Since Γ_1 acts on W by left translations and has finite index in Γ, the projection $p(\Gamma)$ is finite. $\square$

Continuing our proof of the assertion (1), let us denote $\Phi = p(\Gamma)$ and $\Gamma^* = \ker(p)$. Clearly, Γ^* is a discrete subgroup of $\mathcal{N}$ and has a finite index in Γ since Φ is finite.

Now we define $\mathcal{N}_\Gamma$ to be the connected Lie subgroup of the nilpotent group $\mathcal{N}$ that contains the discrete group Γ^* as a lattice. For any elements $\gamma = (a, A) \in \Gamma$ and $\gamma_0 = (b, 1) \in \Gamma^*$, the conjugation $\gamma\gamma_0\gamma^{-1} = (a \cdot A(b) \cdot a^{-1}, 1)$ belongs to Γ^* as Γ^* is normal in Γ. If we set $\tau_a(x) = axa^{-1}$, then the automorphism $\tau_a \circ A$ preserves Γ^*. Since Γ^* is a lattice in $\mathcal{N}_\Gamma$, the automorphism $\tau_a \circ A$ also preserves $\mathcal{N}_\Gamma$. So the map

$$\Gamma \times \mathcal{N}_\Gamma \to \mathcal{N}_\Gamma, \quad \bigl((a, A), v\bigr) \mapsto \tau_a \circ A(v)$$

defines an action of Γ on $\mathcal{N}_\Gamma$. We form the semi-direct product $\mathcal{N}_\Gamma \rtimes \Gamma$ with respect to this action and define

$$K = \bigl\{\bigl(a^{-1}, (a, 1)\bigr) \in \mathcal{N}_\Gamma \rtimes \Gamma\colon\ (a, 1) \in \Gamma^*\bigr\} \subset \mathcal{N}_\Gamma \rtimes \Gamma.$$

It is easy to check that the so-defined K is a normal subgroup of $\mathcal{N}_\Gamma \rtimes \Gamma$. Now we define two maps i and π where

$$i : \mathcal{N}_\Gamma \to (\mathcal{N}_\Gamma \rtimes \Gamma)/K, \quad i(v) = \bigl(v, (1, 1)\bigr)K \quad \text{for } v \in \mathcal{N}_\Gamma$$

and

$$\pi : (\mathcal{N}_\Gamma \rtimes \Gamma)/K \to \Phi, \quad \pi\bigl(v, (a, A)\bigr) = A.$$

Then we get a short exact sequence

$$1 \longrightarrow \mathcal{N}_\Gamma \xrightarrow{i} (\mathcal{N}_\Gamma \rtimes \Gamma)/K \xrightarrow{\pi} \Phi \longrightarrow 1.$$

According to Proposition 3.2, this sequence splits. So there is a homomorphism $s:\Phi\to(\mathcal{N}_\Gamma\rtimes\Gamma)/K$ such that $\pi\circ s$ is the identity id_Φ.

For each $A\in\Phi$, we fix an element $(f(A),(g(A),A))$ in $\mathcal{N}_\Gamma\rtimes\Gamma$ that represents $s(A)$, where $f(A)\in\mathcal{N}_\Gamma$ and $g(A)\in\mathcal{N}$. So we have

$$s(A)=\big(f(A),\big(g(A),A\big)\big)K$$

where $f:\Phi\to\mathcal{N}_\Gamma$ and $g:\Phi\to\mathcal{N}$ are two maps. Since s is a homomorphism, $s(AB)=s(A)s(B)$ for all $A,B\in\Phi$. Thus

$$\big(f(AB),\big(g(AB),AB\big)\big)K=\big(f(A)g(A)A\big(f(B)\big)\big(g(A)\big)^{-1},\big(g(A)A\big(g(B)\big),AB\big)\big)K.$$

Therefore there exists an element $c\in\Gamma^*$ such that

$$\big(f(AB),\big(g(AB),AB\big)\big)\big(c^{-1},(c,1)\big)=\big(f(A)g(A)A\big(f(B)\big)g(A)^{-1},\big(g(A)A\big(g(B)\big),AB\big)\big).$$

It follows that

$$g(A)A\big(g(B)\big)=g(AB)AB(c) \tag{4.1}$$

and

$$f(A)g(A)A\big(f(B)\big)g(A)^{-1}=f(AB)g(AB)AB\big(c^{-1}\big)g(AB)^{-1}. \tag{4.2}$$

Equalities (4.1) and (4.2) combine to give:

$$f(A)g(A)A\big(f(B)\big)=f(AB)g(AB)A\big(g(B)^{-1}\big),$$

which implies

$$g(AB)^{-1}f(AB)^{-1}=A\big(g(B)^{-1}f(B)^{-1}\big)\big(g(A)^{-1}f(A)^{-1}\big). \tag{4.3}$$

Now we define a map $h:\Phi\to\mathcal{N}$ by

$$h(A)=g(A)^{-1}f(A)^{-1}.$$

Then the property (4.3) shows that h is a 1-cocycle. However $H^1(\Phi,\mathcal{N})=0$, so there exists an element $b\in\mathcal{N}$ such that

$$h(A)=A\big(b^{-1}\big)\cdot b \tag{4.4}$$

for all $A\in\Phi$.

On the other hand, for any element $\gamma=(a,A)\in\Gamma$,

$$\pi\big(\big(1,(a,A)\big)K\big)=\pi\big(\big(f(A),\big(g(A),A\big)\big)K\big)=A.$$

Since the sequence $1\to\mathcal{N}_\Gamma\xrightarrow{i}(\mathcal{N}_\Gamma\rtimes\Gamma)/K\xrightarrow{\pi}\Phi\to1$ is exact, there is an element $v_0\in\mathcal{N}_\Gamma$ such that

$$\big(v_0,(1,1)\big)K\cdot\big(f(A),\big(g(A),A\big)\big)K=\big(1,(a,A)\big)K.$$

So there exists $\tilde{c}\in\Gamma^*$ satisfying

$$\big(v_0,(1,1)\big)\cdot\big(f(A),\big(g(A),A\big)\big)=\big(1,(a,A)\big)\cdot\big(\tilde{c}^{-1},(\tilde{c},1)\big).$$

It follows that

$$g(A)=aA(\tilde{c}) \tag{4.5}$$

and

$$v_0 f(A) = aA\big(\tilde{c}^{-1}\big)a^{-1}. \tag{4.6}$$

The properties (4.5) and (4.6) combine to give

$$a^{-1}v_0 = g(A)^{-1} f(A)^{-1},$$

or equivalently,

$$a^{-1}v_0 = h(A). \tag{4.7}$$

Then the properties (4.4) and (4.7) imply that $a^{-1}v_0 = A(b^{-1})b$, and thus

$$baA\big(b^{-1}\big) = bv_0b^{-1}. \tag{4.8}$$

Now consider the group $b\Gamma b^{-1}$. Since the group $b\Gamma^* b^{-1}$ is normal in $b\Gamma b^{-1}$ and is a lattice in $b\mathcal{N}_\Gamma b^{-1}$, the conjugation action of $b\Gamma b^{-1}$ on $b\Gamma^* b^{-1}$ induces an action of $b\Gamma b^{-1}$ on the connected subgroup $b\mathcal{N}_\Gamma b^{-1}$. As before, this action is defined as follows:

$$b\Gamma b^{-1} \times b\mathcal{N}_\Gamma b^{-1} \to b\mathcal{N}_\Gamma b^{-1}, \quad \big((a', A'), v'\big) \mapsto \tau_{a'} \circ A'(v'), \tag{4.9}$$

where $\tau_{a'}(x) = a'x(a')^{-1}$.

For any element $\gamma = (a, A) \in \Gamma$, we have that $b\gamma b^{-1} = (baA(b^{-1}), A)$. So

$$\tau_{baA(b^{-1})} \circ A\big(b\mathcal{N}_\Gamma b^{-1}\big) = b\mathcal{N}_\Gamma b^{-1}.$$

However, due to (4.8) we have that $baA(b^{-1}) \in b\mathcal{N}_\Gamma b^{-1}$, so it implies the A-invariantness, $A(b\mathcal{N}_\Gamma b^{-1}) = b\mathcal{N}_\Gamma b^{-1}$. Now, for any element $b\gamma b^{-1} = (baA(b^{-1}), A)$ in $b\Gamma b^{-1}$, we see that A preserves $b\mathcal{N}_\Gamma b^{-1}$, and $baA(b^{-1})$ belongs to $b\mathcal{N}_\Gamma b^{-1}$. It shows that $b\gamma b^{-1}$ preserves $b\mathcal{N}_\Gamma b^{-1}$, which finishes the proof of the first claim (1) in Theorem 4.1.

To prove the second claim (2) in Theorem 4.1, we see that $b\mathcal{N}_\Gamma b^{-1}/b\Gamma^* b^{-1}$ is compact because $b\Gamma^* b^{-1}$ is a lattice in $b\mathcal{N}_\Gamma b^{-1}$. Therefore $b\mathcal{N}_\Gamma b^{-1}/b\Gamma b^{-1}$ is compact as well.

Since the third assertion (3) is clear, the proof of the whole Theorem 4.1 is complete. □

5. Discrete isometric actions nearby parabolic fixed points

In studying pinched negatively curved spaces, one of the most important tools is given by the Margulis Lemma and the so-called thick-thin decomposition of negatively curved orbifolds/manifolds, see [12,15, 33]. Such orbifolds are quotients $M = X/G$ of Hadamard manifolds X by discrete isometric actions of their fundamental groups $\pi_1^{orb} \cong G \subset \operatorname{Isom} X$. Adding the induced discrete action of G in some domain at infinity ∂X, we obtain a partial closure $M(G)$ of that orbifold M. More precisely, let $\Lambda(G) \subset \partial X$ be the *limit set* of a discrete group $G \subset \operatorname{Isom} X$, that is the set of accumulation points of (any) orbit $G(y)$, $y \in X$. A discrete group G is called *elementary* if its limit set $\Lambda(G)$ consists of at most two points. Such an elementary group G may be either finite, or hyperbolic one that setwise preserves an infinite geodesic in X, or parabolic one whose limit set is an unique fixed point of G, $\Lambda(G) = \{p\} = \operatorname{fix}(G)$. The complement of $\Lambda(G)$ in ∂X is called the *discontinuity set* $\Omega(G)$. Then, for any discrete group $G \subset \operatorname{Isom} X$, we set

$$M(G) = \big(X \cup \Omega(G)\big)/G. \tag{5.1}$$

Let ε be any positive number less than $\varepsilon(n,a,b)$, the Margulis constant in dimension n depending on the pinching constants $a, b > 0$. For a given discrete group $G \subset \operatorname{Isom} X$ and its orbifold $M = X/G$, the ε-thin part $\operatorname{thin}_\varepsilon(M)$ is defined as follows:

$$\operatorname{thin}_\varepsilon(M) = \bigl\{x \in X\colon\ G_\varepsilon(x) = \bigl\langle g \in G\colon\ d\bigl(x, g(x)\bigr) < \varepsilon\bigr\rangle \text{ is infinite}\bigr\}/G. \tag{5.2}$$

The thick part $\operatorname{thick}_\varepsilon(M)$ of an orbifold M is defined as the closure of the complement to the thin part, $\operatorname{thin}_\varepsilon(M) \subset M$.

As a consequence of the Margulis Lemma, there is the following description of the thin part of M [12,15]:

Theorem 5.1. *Let $G \subset \operatorname{Isom} X$ be a discrete group and ε, $0 < \varepsilon < \varepsilon(n,a,b)$, be chosen. Then the ε-thin part $\operatorname{thin}_\varepsilon(M)$ of $M = X/G$ is a disjoint union of its connected components, and each such component has the form $T_\varepsilon(\Gamma)/\Gamma$ where Γ is a maximal infinite elementary subgroup of G. Here, for each such elementary subgroup $\Gamma \subset G$, the connected component* (*Margulis region*)

$$T_\varepsilon = T_\varepsilon(\Gamma) = \bigl\{x \in X\colon\ \Gamma_\varepsilon(x) = \bigl\langle g \in \Gamma\colon\ d\bigl(x, \gamma(x)\bigr) < \varepsilon\bigr\rangle \textit{ is infinite}\bigr\} \tag{5.3}$$

is precisely invariant with respect to the subgroup Γ in G:

$$\Gamma(T_\varepsilon) = T_\varepsilon, \quad g(T_\varepsilon) \cap T_\varepsilon = \emptyset \quad \textit{for any } g \in G \backslash \Gamma. \tag{5.4}$$

We note that in the real hyperbolic case of dimension 2 and 3, a Margulis region T_ε in (5.3) with parabolic stabilizer $\Gamma \subset G$ can be taken as a horoball neighborhood centered at the parabolic fixed point p, $\Gamma(p) = p$. It is not true in general due to Apanasov's construction [4] in real hyperbolic spaces of dimension at least 4. As we discussed it in [10], this construction works in complex hyperbolic spaces $\mathbb{H}^n_{\mathbb{C}}$ as well as in other rank one symmetric spaces, and probably in any pinched Hadamard manifold X.

However, we are able to give a description of parabolic Margulis regions for any discrete group $G \subset \operatorname{Isom} X$ by applying the structural Theorem 4.1 to actions of parabolic groups.

Namely, let $\Gamma \subset G$ be a discrete parabolic subgroup. We may view X from the fixed point $p \in \partial X$ in the way we have used in Section 2 to define the upper half-space model for X. Namely, by using the foliation of X by horospheres X_t centered at p, we identify $\overline{X} \backslash \{p\}$ and $\mathcal{N} \times [0,\infty)$ as in (2.3), where $X_t \cong \mathcal{N}$ is a connected, simply connected Lie group with a compact automorphism group C. Since the parabolic group Γ acts on each horosphere X_t centered at the fixed point p as a discrete subgroup of $\mathcal{N} \rtimes C$, we can apply Theorem 4.1. So there exist a Γ-invariant connected subspace $\sigma \subset \partial X \backslash \{p\} \cong \mathcal{N}$ where Γ acts co-compactly and on which a finite index subgroup $\Gamma^* \subset \Gamma$ acts freely by left translations. In fact, σ is a translate of a connected Lie subgroup $\mathcal{N}_\Gamma$ of $\partial X \backslash \{p\} \cong \mathcal{N}$. Now we define the subspace $\tau \subset X$ to be spanned by σ and all geodesics $(z, p) \subset X$ connecting $z \in \sigma$ to the parabolic fixed point p. Let τ_t be the "half-plane" in τ of a height $t > 0$, that is the part of τ whose last horospherical coordinate is at least t, see Fig. 1.

Lemma 5.2. *Let $G \subset \operatorname{Isom} X$ be a discrete group in a pinched Hadamard manifold X having N-property and p a parabolic fixed point of G. Let T_ε be a Margulis region for p as given in* (5.3) *and let τ_t be the half-plane defined as above. Then for any δ, $0 < \delta < \varepsilon/2$, there exists a positive number $t > 0$ such that the Margulis region T_ε contains the δ-neighborhood $N_\delta(\tau_t)$ of the half-plane τ_t.*

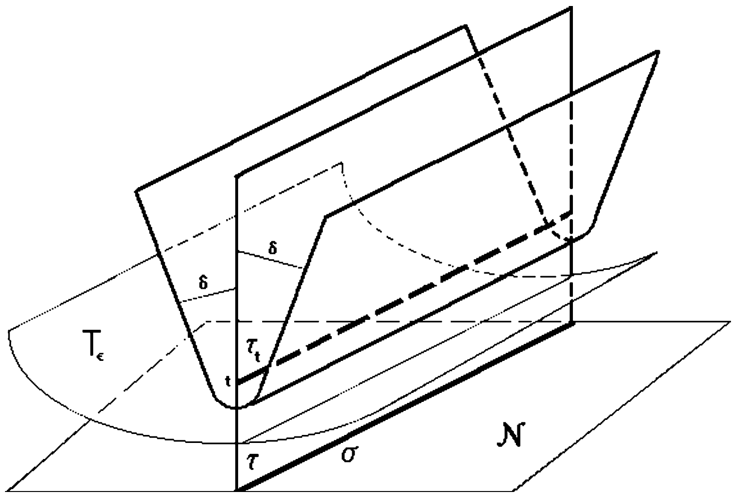

Fig. 1. Half-plane τ_t and its δ-neighborhood.

Proof. Let $\Gamma \subset G$ be the maximal parabolic subgroup fixing a given parabolic fixed point $p \in \partial X$. Since Γ preserves horospheres centered at p and the subspace $\sigma \subset \partial X \backslash \{p\}$, it preserves the boundary $\partial\tau_t$ of each half-plane τ_t.

Let us consider positive numbers δ and ε' so that $2\delta + \varepsilon' < \varepsilon$. Due to (2.1), we have the exponential decrease of distances on horospheres in X of height t as t goes to $+\infty$. On the other hand, due to Theorem 4.1, infinite order elements $\gamma \in \Gamma$ act on the boundary $\partial\tau_t$, $t > 0$, as virtual translations, and the quotient $\partial\tau_t/\Gamma$ is compact. Therefore there exist some height $t_{\varepsilon'}$ such that

$$\partial\tau_t \subset T_{\varepsilon'}(\Gamma) \subset T_{\varepsilon'}(G) = T_{\varepsilon'} \quad \text{for all } t > t_{\varepsilon'}.$$

It immediately follows that the same is true for the whole half-plane:

$$\tau_t \subset T_{\varepsilon'}(\Gamma) \subset T_{\varepsilon'}. \tag{5.5}$$

Now, for any $x \in N_\delta(\tau_t)$ with $t > t_{\varepsilon'}$, we have a δ-close point $x_0 \in \tau_t$, $d(x, x_0) < \delta$. Due to (5.5), there is an infinite order element $\gamma \in \Gamma$ such that $d(x_0, \gamma(x_0)) < \varepsilon'$. It implies:

$$d\big(x, \gamma(x)\big) \leqslant d(x, x_0) + d\big(x_0, \gamma(x_0)\big) + d\big(\gamma(x_0), \gamma(x)\big) < 2\delta + \varepsilon' < \varepsilon,$$

which shows that the point x and thus the whole δ-neighborhood $N_\delta(\tau_t)$ belong to the Margulis region T_ε. $\square$

Now we can (negatively) answer the question of *whether a parabolic fixed point of a discrete group* $G \subset \operatorname{Isom} X$ *may also be its conical limit point.*

Here a limit point $z \in \Lambda(G)$ is called a *conical limit point* of a discrete group $G \subset \operatorname{Isom} X$ if, for some (and hence every) geodesic ray $\ell \subset X$ ending at z, there is a compact set $K \subset X$ such that $g(\ell) \cap K \neq \emptyset$ for infinitely many elements $g \in G$.

This definition is equivalent to a possibility to approximate the limit point $z \in \Lambda(G)$ by a G-orbit $\{g_i(x)\}$ of a point $x \in X$ inside a tube (cone) in X with vertex $z \in \partial X$. Representing the negatively curved space X by its ball model, one can see that approximating cone with vertex at z may be however tangent in some directions to the boundary sphere ∂X, in an appropriate metric in the ball; see Koranyi [29,30] and, for complex hyperbolic space, Kamiya [27]. Applying an argument originally due to Beardon and Maskit [13], one can use the following equivalent definition of conical limit points:

Lemma 5.3. *A point $z \in \Lambda(G)$ is a conical limit point of a discrete group $G \subset \operatorname{Isom} X$ in a negatively curved space X if and only if, for every geodesic ray $\ell \subset X$ ending at z and for every $\delta > 0$, there is a point $x \in X$ and a sequence of distinct elements $g_i \in G$ such that the orbit $\{g_i(x)\}$ approximates z inside the δ-neighborhood $N_\delta(\ell)$ of the ray ℓ.*

Proof. To see the equivalence of two definitions, it is enough to find a sequence $g_i(x)$ that approximates z δ-closely to a given geodesic ray ℓ. Due to the first definition, we start with infinitely many non-empty intersections $g_i(\ell) \cap K$ with a compact K. Since they accumulate somewhere in K, we may substitute K with another compact set $K_\delta \subset X$ with diameter less than δ. Thus $\ell \cap g_i^{-1}(K_\delta) \neq \emptyset$, and we have that $g_i^{-1}(K_\delta) \subset N_\delta(\ell)$ for all i. It implies that for any $x \in K_\delta$, its distinct images $g_i^{-1}(x)$ lie in $N_\delta(\ell)$ and approximate z because of discreteness of G.

Conversely, let us assume that a limit point $z \in \Lambda(G)$ satisfies the condition of lemma. Then we may take the closed δ-neighborhood $N_\delta(x)$ as the compact set $K \subset X$ in the first definition of conical limit points. Indeed, since all points in the orbit $\{g_i(x)\}$ lie in the δ-neighborhood $N_\delta(\ell)$ of the ray ℓ, each δ-neighborhood $N_\delta(g_i^{-1}(\ell))$ contains the point x. It shows that $K \cap g_i^{-1}(\ell) \neq \emptyset$ for all i, so the point z satisfies the first definition of conical limit points. □

There are another (equivalent) definitions of conical limit points, see [13,15]. One of them is even intrinsic to the action of the group G on the limit set $\Lambda(G)$. Namely, $z \in \Lambda(G)$ is a conical limit point if there is a sequence $\{g_i\}$ of distinct elements of G such that, for any other limit point $y \in \Lambda(G)\backslash\{z\}$, the sequence of pairs $(g_i^{-1}(z), g_i^{-1}(y))$ lies in a compact subset of $(\Lambda(G) \times \Lambda(G))\backslash\Delta(\Lambda)$, where $\Delta(\Lambda) = \{(x,x)\colon x \in \Lambda(G)\}$.

Proposition 5.4. *Let $G \subset \operatorname{Isom} X$ be a discrete group in a pinched Hadamard manifold X having N-property. Then any parabolic fixed point of G cannot be its conical limit point.*

Proof. Let $p \in \partial X$ be a parabolic fixed point of the group G and $\Gamma \subset G$ the maximal discrete parabolic subgroup fixing p. As before in Lemma 5.2, we may view X from the point p at infinity by using horospherical coordinates in Section 2. Then applying Theorem 4.1, we again have a γ-invariant connected subspace $\sigma \subset \partial X\backslash\{p\}$ where Γ acts co-compactly and on which a finite index subgroup $\Gamma^* \subset \Gamma$ acts freely by left translations. As before, let $\tau \subset X$ be the subspace spanned by σ and p. Furthermore, due to Lemma 5.2, there are positive numbers δ and t so that the δ-neighborhood $N_\delta(\tau_t)$ of the half-plane τ_t is contained in the parabolic Margulis region T_ε at p.

Now suppose that the point p is also a conical limit point of G. Then, as we have shown in Lemma 5.3, for a geodesic ray $\ell \subset \tau_t$ tending to p, there must exist a point $x \in X$ and a sequence of distinct elements $g_i \in G$ such that the sequence $g_i(x)$ tends to p inside of δ-neighborhood $N_\delta(\ell)$ of the ray ℓ. However, due to Lemma 5.2, $N_\delta(\ell) \subset N_\delta(\tau_t) \subset T_\varepsilon$. Since the Margulis (parabolic) region T_ε is precisely invariant for the subgroup $\Gamma \subset G$ (Theorem 5.1), it follows that all elements g_i belong in fact to the parabolic subgroup Γ. Hence all g_i preserve each horosphere X_t centered at p. Using compactness of $\partial\tau_t/\Gamma$, we see then that all points $g_i(x)$ must lie in a compact part of $N_\delta(\tau_t)$ and hence cannot approach the limit point p. This contradiction completes the proof. □

6. Parabolic cusps and geometrical finiteness

Here we shall apply the structural Theorem 4.1 to clarify the structure of cusp ends of geometrically finite pinched negatively curved manifolds/orbifolds, in particular those of locally symmetric spaces of rank 1, where the theory is much less well developed than that in the real and complex hyperbolic spaces.

Though there are many equivalent definitions of geometrical finiteness for manifolds with variable curvature [15], examples of discrete parabolic groups acting on complex hyperbolic space due to [10, 22] suggest that no elegant formulation of geometrical finiteness involving finite-sided polyhedra exists. Also the topological definition based on the properties of ends of the orbifold $M(G)$ given in (5.1) is not so far as applicable as in the case of real or complex hyperbolic manifolds, see [3,7,10]. In particular, the topological finiteness of geometrically finite manifolds still remains unclear despite the fact that geometrically finite groups are finitely generated [15]. It seems to us, a reason for this has its origin in a non-geometric definition of parabolic cusp points used in variable curvature spaces [15]. The aim of this section is to provide a new insight to that crucial notion, which in particular allow us to prove the fact that geometrically finite groups are in fact finitely presented.

Due to a definition in [15], a parabolic fixed point $p \in \partial X$ of a discrete group $G \subset \operatorname{Isom} X$ in a pinched negatively curved space X is called a *cusp point* if the quotient $(\Lambda(G) \setminus \{p\})/G_p$ of the limit set of G by the action of the parabolic stabilizer $G_p = \{g \in G\colon\ g(p) = p\}$ is compact.

This leads to a definition (GF1, originally due to Beardon and Maskit [13]) of *geometrically finite* discrete groups $G \subset \operatorname{Isom} X$ (and their negatively curved orbifolds $M = X/G$) as those whose limit set $\Lambda(G) \subset \partial X$ entirely consists of conical limit points and parabolic cusps.

Another definition of geometrical finiteness (GF2, originally due to Marden [32]) is that the quotient $M(G)$ in (5.1) has only finitely many of topological ends and each of them can be identified with the end of $M(\Gamma)$, where Γ is a maximal parabolic subgroup of G.

Additional two definitions of geometrical finiteness are originally due to Thurston [39]):

(GF3) The thick part of the minimal convex retract (= convex core) $C(G)$ of X/G is compact.
(GF4) For some $\varepsilon > 0$, the uniform ε-neighborhood of the convex core $C(G) \subset X/G$ has finite volume, and there is a universal bound on the orders of finite subgroups in G.

Theorem 6.1 [15]. *Let X be a pinched Hadamard manifold. Then the four definitions* GF1, GGF2, GGF3 *and* GGF4 *of geometrical finiteness for a discrete group $G \subset \operatorname{Isom} X$ are all equivalent.*

Now we shall give a new geometric definition of parabolic cusp points (cusp ends) in pinched Hadamard manifolds X having N-property. Namely, suppose a point $p \in \partial X$ is a parabolic fixed point of a discrete group $G \subset \operatorname{Isom} X$ and let $\Gamma = G_p$ be the stabilizer of p in G (i.e., a maximal parabolic subgroup in G). Taking horospherical coordinates on X with respect to the point p at infinity (as in (2.3)), we can regard this stabilizer as $\Gamma \subset \mathcal{N} \rtimes C$ where C is a compact automorphism group of the connected Lie group $\mathcal{N}$ representing horospheres in X. Let $\rho_{\mathcal{N}}$ be a left invariant metric on the nilpotent Lie group $\mathcal{N}$, which is $\mathcal{N} \rtimes C$-invariant (due to compactness of C, see Remark 4.2). Also, by $\mathcal{N}_\Gamma \subseteq \mathcal{N}$ we denote a minimal connected subgroup of the nilpotent group $\mathcal{N} \cong \partial X \backslash \{p\}$ preserved by the parabolic stabilizer Γ, and where the group Γ acts cocompactly, see Theorem 4.1.

Definition 6.2. Given a positive number r and a parabolic fixed point $p \in \partial X$ of a discrete group $G \subset \operatorname{Isom} X$ with stabilizer $\Gamma = G_p \subset G$, the set

$$U_{p,r} = \left\{x \in \partial X \backslash \{p\} \cong \mathcal{N}\colon\ \rho_{\mathcal{N}}(x, \mathcal{N}_\Gamma) \geqslant 1/r\right\} \tag{6.1}$$

is called a *standard cusp neighborhood* of radius $r > 0$ at p, provided it is precisely invariant with respect to the stabilizer Γ in G:

$$\begin{aligned} &\gamma(U_{p,r}) = U_{p,r} && \text{for } \gamma \in \Gamma = G_p, \\ &g(U_{p,r}) \cap U_{p,r} = \emptyset && \text{for } g \in G \backslash G_p. \end{aligned}$$

Lemma 6.3. *Let $p \in \partial X$ be a parabolic fixed point of a discrete group $G \subset \operatorname{Isom} X$ in a pinched Hadamard manifold X having N-property. Then p is a parabolic cusp point if and only if it has a standard cusp neighborhood $U_{p,r}$.*

Proof. As before, let $\Gamma \subset G$ be the parabolic stabilizer of a given parabolic fixed point p. Then Γ preserves $\mathcal{N}_\Gamma \subseteq \mathcal{N} \cong \partial X \backslash \{p\}$, the minimal connected Γ-invariant subspace of $\mathcal{N}$ given by Theorem 4.1. If p has a standard cusp neighborhood $U_{p,r} \subset \mathcal{N}$ then the limit set $\Lambda(G)$ must lie in its complement $\partial X \backslash U_{p,r}$ due to the condition of its precise Γ-invariantness. Hence $\Lambda(G)\backslash\{p\}/\Gamma$ is compact because $\mathcal{N}_\Gamma/\Gamma$ is compact due to Theorem 4.1. The converse statement follows from Bowditch's arguments in the proof [15] of Theorem 6.1. $\square$

Now we can extend the $(\mathcal{N} \rtimes C)$-invariant metric $\rho_{\mathcal{N}}$ at infinity $\partial X \backslash \{p\} \cong \mathcal{N}$ to a $(\mathcal{N} \rtimes C)$-invariant metric ρ in $\overline{X} \backslash \{p\} = \mathcal{N} \times [0, \infty)$. It can be done in many ways, similarly to the extension of the Cygan metric in the Heisenberg group representing the infinity of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, and more generally for rank one symmetric spaces, see (2.10) in Section 2. Namely, as such a (simplest) metric ρ that extends the $(\mathcal{N} \rtimes C)$-invariant metric $\rho_{\mathcal{N}}$, we can take the $(\mathcal{N} \rtimes C)$-invariant product metric ρ in $\overline{X} \backslash \{p\} = \mathcal{N} \times [0, \infty)$ where $[0, \infty)$ has a metric commensurable with the square root of the Euclidean one.

Remark 6.4. While this paper has been completed, Kamiya introduced the first author with a paper by Korányi [30] which recalls the notion (originally introduced in his old paper [29]) of conical approach to points p at infinity of rank one symmetric spaces and related metrics on horospheres centered at p. In particular, in gauge terms (see (2.3) in [30]) on Lie algebras $\mathfrak{n} \subset \mathfrak{g}$ of Lie groups $\mathcal{N} \subset \mathcal{G} = \operatorname{Isom} X$, it presents another construction of a left-invariant metric on a nilpotent Carnot group $\mathcal{N}$ associated with a symmetric space of rank one.

Now, with respect to such an extended metric ρ in X, we define standard cusp X-neighborhoods of radius $r > 0$ as:

$$\widehat{U}_{p,r} = \left\{x \in \overline{X} \backslash \{p\} \cong \mathcal{N} \times [0, \infty)\colon\ \rho(x, \mathcal{N}_\Gamma) \geqslant 1/r\right\}, \tag{6.2}$$

whose boundaries at infinity coincide with the standard cusp neighborhoods in the Carnot group, $U_{p,r} \subset \partial X \backslash \{p\}$.

Furthermore, due to Lemma 5.2, we can define (a metric on $[0,\infty)$ and) the extension metric ρ so that $\widehat{U}_{p,r}$ are precisely G_p-invariant, as $U_{p,r}$ are.

For a given discrete group $\Gamma \subset \mathcal{N} \rtimes C \subset \operatorname{Isom} X$, the quotient space with respect to its isometric action, $(X \cup \partial X \backslash \{\infty\})/\Gamma$, has a unique end. We call this end a *standard parabolic end* with a $(X, \operatorname{Isom} X)$-geometry. It is clear that neighborhoods of a standard parabolic end may be taken as $\widehat{U}_{\infty,r}/\Gamma$, $r > 0$.

Applying the above point of view on cusp ends, Lemma 6.3 and Theorem 6.1, we see that in general case of a cusp point $p \in \partial X$ of a geometrically finite discrete group $G \subset \operatorname{Isom} X$, the family $E_p = \{\widehat{U}_{p,r}/G_p\}$ of closed suborbifolds in the orbifold $M(G)$ naturally defines the cusp end of $M(G)$ identified by G-orbit of the parabolic cusp point p. It is isometric to a standard cusp end, actually to the end of $M(G_p)$.

We may represent a standard cusp X-neighborhood $\widehat{U}_{p,r_0}$ at a cusp point p of a discrete group $G \subset \operatorname{Isom} X$ as the product

$$\widehat{U}_{p,r_0} = S_{p,r_0} \times (0, r_0], \tag{6.3}$$

if we foliate $\widehat{U}_{p,r_0}$ by subsets $S_{p,r},\ 0 < r \leqslant r_0$, of the form:

$$S_{p,r} = \big\{x \in X \cup \partial X \backslash \{p\} \cong \mathcal{N} \times [0,\infty):\ \rho(x, \mathcal{N}_{G_p}) = 1/r\big\}. \tag{6.4}$$

Since each set $S_{p,r}$ is G_p-invariant, we see that the standard cusp X-neighborhood $\widehat{U}_{p,r_0}/G_p \subset M(G)$ of the cusp end E_p of the orbifold $M(G)$ is the product $(S_{p,r_0}/G_p) \times (0,1]$. Furthermore, due to compactness of the automorphism group C of the nilpotent group $\mathcal{N}$, this foliation of a standard cusp X-neighborhood $\widehat{U}_{p,r_0}$ by G_p-invariant sets $S_{p,r}$ defines a G_p-equivariant retraction

$$R_p : \widehat{U}_{p,r_0} \to \mathcal{N}_{G_p}. \tag{6.5}$$

This retraction shows topological finiteness of ends of noncompact orbifolds $\mathcal{N}/\Gamma$ for discrete parabolic groups $\Gamma \subset \mathcal{N} \rtimes C$ (and, with a little bit more work, existence of a vector bundle structure on them, compare [10, Theorem 4.1]) as well as topological finiteness of cusp ends of $(X, \operatorname{Isom} X)$-orbifolds locally modeled on a pinched Hadamard manifolds X with N-property. So, due to Theorem 4.1, all those ends have the homotopy type of closed virtually nilpotent orbifolds $\mathcal{N}_\Gamma/\Gamma$. This, together with Theorem 6.1, completes the proof of the following fact:

Theorem 6.5. *Let X be a pinched Hadamard manifold having N-property with a connected nilpotent Lie group $\mathcal{N}$ and its compact automorphism group $C \subset \operatorname{Aut}\mathcal{N}$. Then, for any geometrically finite discrete group $G \subset \operatorname{Isom} X$, the orbifold $M = X/G$ is topologically finite, that is, M is orbifold-homeomorphic to the interior of a compact orbifold with boundary obtained from $M(G)$ by gluing to its ends closed virtually nilpotent orbifolds of the form $\mathcal{N}_\Gamma/\Gamma$ where $\Gamma \subset \mathcal{N} \rtimes C$ is a parabolic discrete group.*

It immediately implies:

Corollary 6.6. *Let X, $\mathcal{N}$ and $C \subset \operatorname{Aut}\mathcal{N}$ be as in Theorem* 6.5, *which includes all rank one symmetric spaces. Then all discrete parabolic groups $\Gamma \subset \mathcal{N} \rtimes C$ as well as geometrically finite groups $G \subset \operatorname{Isom} X$ are finitely presented.*

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